OFDM under AWGN and Static Channel

BPSK & 16-QAM Simulation

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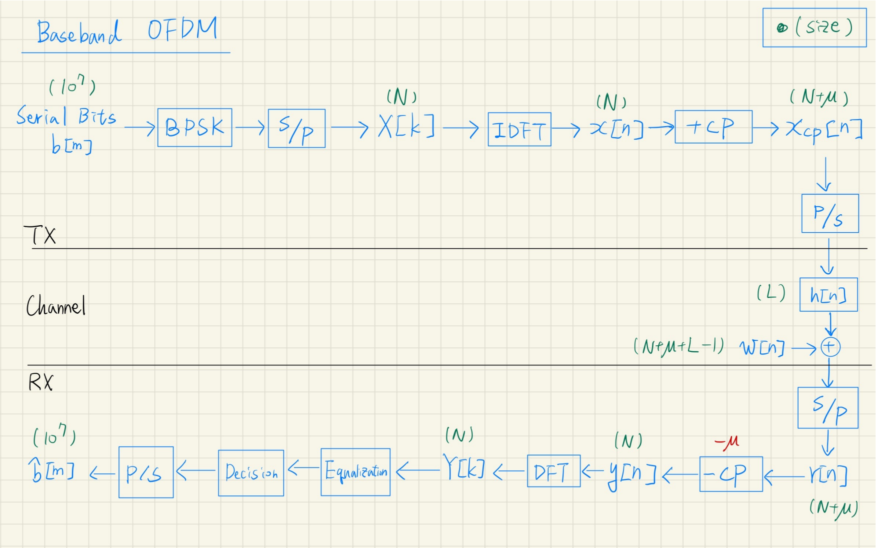
Overview

Block Diagram



Model

The equivalent time-domain baseband received signal is

$$y[n] = h[n] \circledast x[n] + w[n], \quad n = 0, 1, \dots, N-1.$$

Taking the DFT of both sides gives

$$\boxed{Y_k = H_k X_k + W_k, \quad k = 0, 1, \dots, N-1.}$$

where

- $X_k \in \{+1, -1\}$
- $\mathcal{E}_s = E\{X_k^2\} = \frac{1^2+1^2}{2} = 1$
- $H_k \in \mathbb{C}$
- $W_k \sim \mathcal{CN}(0, N_0)$
  $W_k = W_{kI} + jW_{kQ}, \quad W_{kI}, W_{kQ} \sim \mathcal{N}(0, \frac{N_0}{2})$

Demodulation (Equalization & Decision)

We want to demodulate X_k from Y_k:

$$\begin{aligned} Y_k &= H_k X_k + W_k \\ &= |H_k| e^{j\angle H_k} X_k + W_k, \quad W_k \sim \mathcal{CN}(0, N_0) \\ Y_k e^{-j\angle H_k} &= |H_k| X_k + W_k e^{-j\angle H_k}, \quad W_k e^{-j\angle H_k} \sim \mathcal{CN}(0, N_0) \\ \Rightarrow \Re\{Y_k e^{-j\angle H_k}\} &= |H_k| X_k + \Re\{W_k e^{-j\angle H_k}\}, \quad \Re\{W_k e^{-j\angle H_k}\} \sim \mathcal{N}(0, \frac{N_0}{2}) \\ \Im\{Y_k e^{-j\angle H_k}\} &= \Im\{W_k e^{-j\angle H_k}\}, \quad \Im\{W_k e^{-j\angle H_k}\} \sim \mathcal{N}(0, \frac{N_0}{2}) \end{aligned}$$

Note that the imaginary part contains only noise, so we focus on the real part:

$$\tilde{Y}_k = \Re\{Y_k e^{-j\angle H_k}\} = |H_k| X_k + \tilde{W}_k, \quad \tilde{W}_k = \Re\{W_k e^{-j\angle H_k}\} \sim \mathcal{N}(0, \frac{N_0}{2})$$

Since $X_k = \pm 1$, the decision threshold is 0. That is,

$$\boxed{\hat{X}_k = \text{sgn}(\tilde{Y}_k) \ \text{,where } \tilde{Y}_k = \Re\{Y_k e^{-j\angle H_k}\} = \Re\{Y_k \frac{H_k^*}{|H_k|}\}}$$

Transmitted SNR

Before demodulation:

$$Y_k = H_k X_k + W_k$$

where $\mathcal{E}_s = E\{X_k^2\}$ and $W_k \sim \mathcal{CN}(0, N_0)$.

Hence, the transmitted SNR is:

$$\boxed{\text{Transmitted SNR} = \frac{E\{X_k^2\}}{Var\{W_k\}} = \frac{\mathcal{E}_s}{N_0}}$$

Note: $\mathcal{E}_s = \mathcal{E}_b$.

Received SNR

After demodulation:

$$\tilde{Y}_k = |H_k| X_k + \tilde{W}_k$$

where $\mathcal{E}_s = E\{X_k^2\}$ and $\tilde{W}_k \sim \mathcal{N}(0, \frac{N_0}{2})$.

Hence, the received SNR is:

$$\text{Received SNR}_k = \frac{E\{(|H_k| X_k)^2\}}{Var\{\tilde{W}_k\}} = \frac{|H_k|^2 E\{X_k^2\}}{N_0/2} = \frac{2|H_k|^2 \mathcal{E}_s}{N_0}$$

$$\text{Average Received SNR} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{2|H_k|^2 \mathcal{E}_s}{N_0} = \frac{2\mathcal{E}_s}{N_0} \frac{1}{N} \sum_{k=0}^{N-1} |H_k|^2$$

If $h[n]$ is normalized, then $\frac{1}{N} \sum_k |H[k]|^2 = \sum_n |h[n]|^2 = 1$. Therefore:

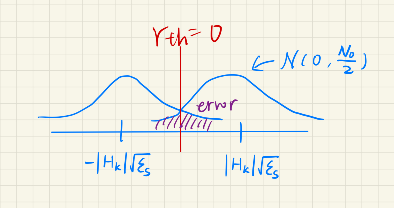
$$\boxed{\text{Average Received SNR} = \frac{2\mathcal{E}_s}{N_0}}$$

BER

$$\tilde{Y}_k = |H_k| X_k + \tilde{W}_k$$

where $\|X_k\| = \sqrt{\mathcal{E}_s}$,
$k = 0, 1, \ldots, N-1$.

Assume equal prior probabilities (i.e. $P(X_k = \pm 1) = \frac{1}{2}$).



The BER for the kth symbol is:

$$P_{b,k} = 2 \times \left(\frac{1}{2} \int_{\frac{|H_k|\sqrt{\mathcal{E}_s}}{\sqrt{N_0/2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right) = Q\left(\frac{|H_k|\sqrt{\mathcal{E}_s}}{\sqrt{N_0/2}} \right) = Q\left(\sqrt{\frac{2|H_k|^2 \mathcal{E}_s}{N_0}} \right)$$

Hence, the average BER is:

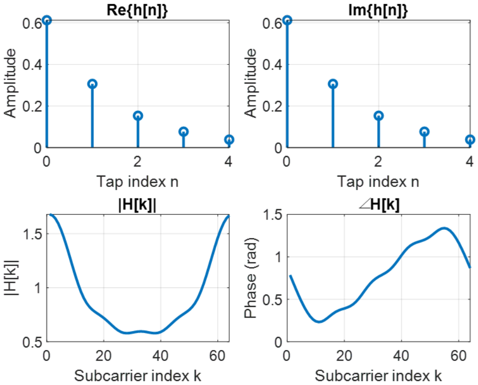
$$\text{BER} = E\{P_{b,k}\} = E\left\{ Q\left(\sqrt{\frac{2|H_k|^2 \mathcal{E}_s}{N_0}} \right) \right\}$$

Parameters

- Number of transmission bits: 64×10^6
- Number of subcarriers: 64
- Length of CP: 16
- Channel length: 5
- Impulse response (before normalization):

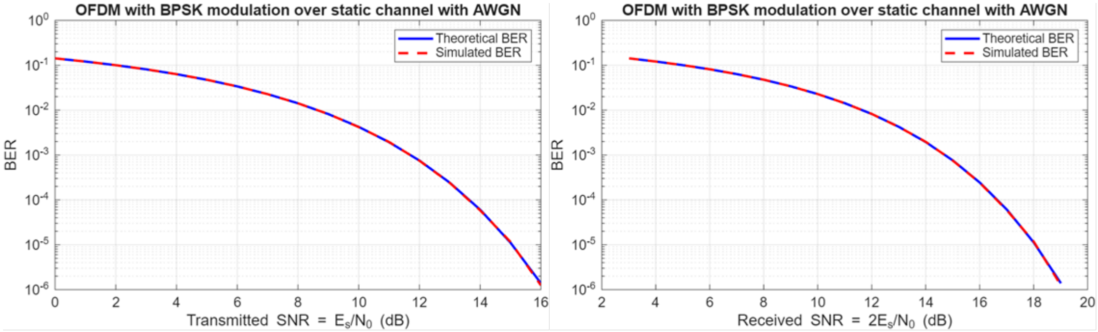
$$h[n] = 0.5^n(1+j),\ n = 0, 1, \dots, 4$$

- Offset of FFT windows: 0



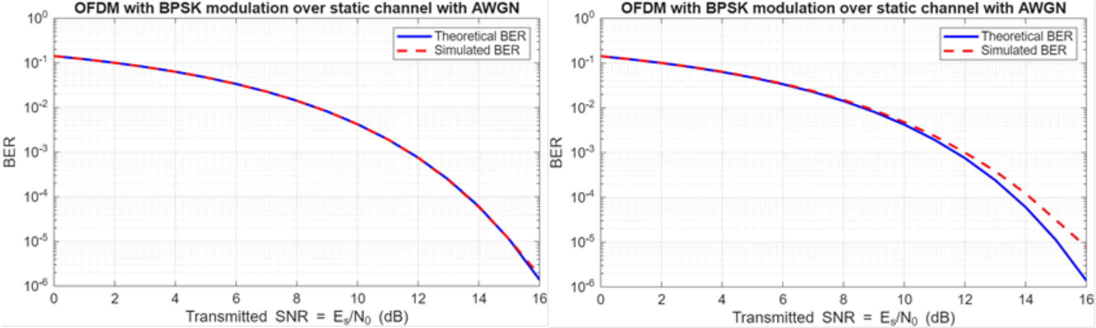
The $h[n]$ in figure is after normalized.

Simulation Results



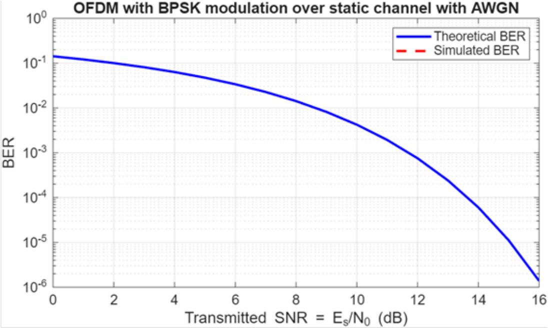
Note that the two graphs differ by 3 dB.

Different Offsets of FFT Windows



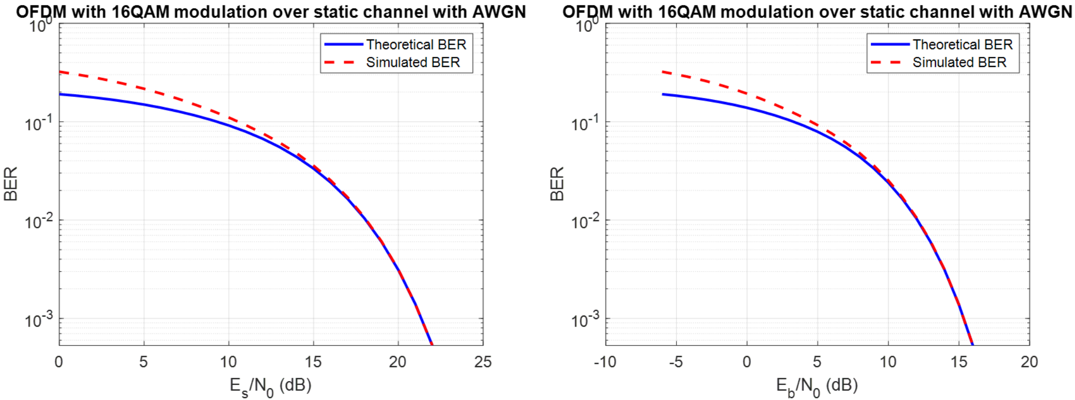
- The first and second figures have offsets of 12 and 16, respectively.
- The maximum offset to avoid ISI can be calculated by:
  Length of CP $-$ Length of Channel $+ 1 = 16 - 5 + 1 = 12$

Zero Noise



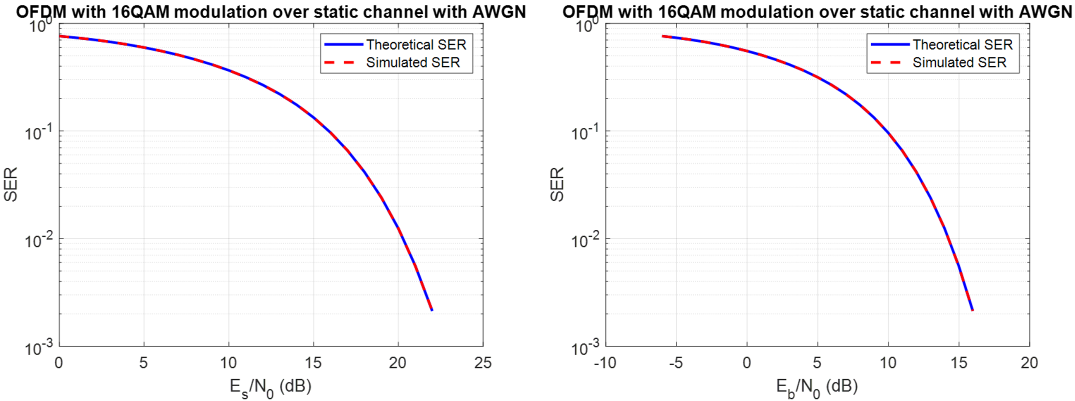
Since there is no noise, the BER is zero.

Simulation of 16-QAM



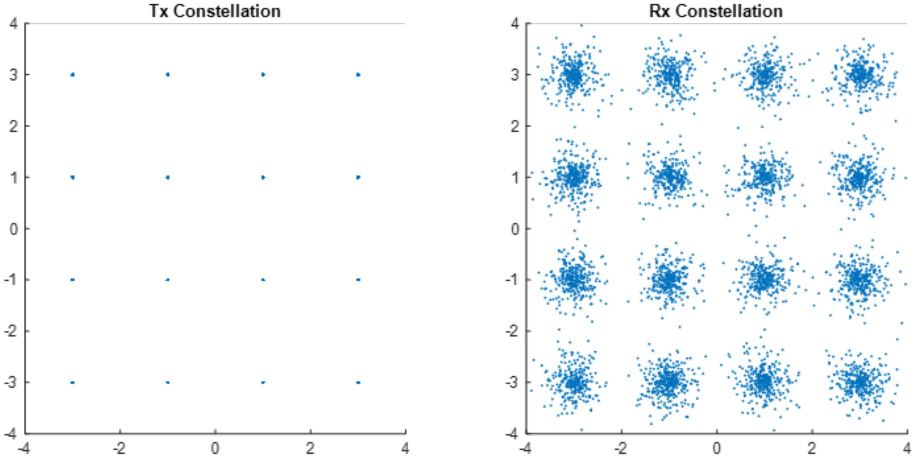
- The simulation parameters are identical to those used for BPSK.
- The theoretical BER is plotted using $BER \approx SER/4$ (Gray coding), which is accurate only at high SNR; hence it deviates at low SNR.

Simulation of 16-QAM



SER curve.

Simulation of 16-QAM



The constellation under AWGN.

Model of 16-QAM

The equivalent time-domain baseband received signal is

$$y[n] = h[n] \circledast x[n] + w[n], \quad n = 0, 1, \dots, N-1.$$

Taking the DFT of both sides gives

$$\boxed{Y_k = H_k X_k + W_k, \quad k = 0, 1, \dots, N-1.}$$

where

- $X_k = I_k + jQ_k, \quad I_k, Q_k \in \{\pm 1, \pm 3\}$
- $\mathcal{E}_s = E\{|X_k|^2\} = E\{I_k^2\} + E\{Q_k^2\} = 2 \times \frac{1^2+1^2+3^2+3^2}{4} = 10$
- $H_k \in \mathbb{C}$
- $W_k \sim \mathcal{CN}(0, N_0)$
  $W_k = W_{kI} + jW_{kQ}, \quad W_{kI}, W_{kQ} \sim \mathcal{N}(0, \frac{N_0}{2})$

Demodulation of 16-QAM

We want to demodulate X_k from Y_k:

$$Y_k = H_k X_k + W_k, \quad W_k \sim \mathcal{CN}(0, N_0)$$

$$\text{By MD rule}: \ \hat{X}_k = \arg\min_{x \in \mathcal{S}_{16}} |Y_k - H_k x|^2 = \arg\min_{x \in \mathcal{S}_{16}} |\frac{Y_k}{H_k} - x|^2$$

$$\tilde{Y}_k = \frac{1}{H_k} Y_k = X_k + \tilde{W}_k, \quad \tilde{W}_k = \frac{1}{H_k} W_k \sim \mathcal{CN}(0, \frac{N_0}{|H_k|^2})$$

$$\Rightarrow \mathfrak{R}\{\tilde{Y}_k\} = I_k + \mathfrak{R}\{\tilde{W}_k\}, \quad \mathfrak{R}\{\tilde{W}_k\} \sim \mathcal{N}(0, \frac{N_0}{2|H_k|^2})$$

$$\mathfrak{I}\{\tilde{Y}_k\} = Q_k + \mathfrak{I}\{\tilde{W}_k\}, \quad \mathfrak{I}\{\tilde{W}_k\} \sim \mathcal{N}(0, \frac{N_0}{2|H_k|^2})$$

Since $I_k, Q_k \in \{\pm 1, \pm 3\}$, the decision thresholds are $-2, 0, 2$ to get $\hat{I}_k, \hat{Q}_k$. Then

$$\boxed{\hat{X}_k = \hat{I}_k + j\hat{Q}_k}$$

Transmitted SNR of 16-QAM

Before demodulation:

$$Y_k = H_k X_k + W_k$$

where $\mathcal{E}_s = E\{|X_k|^2\}$ and $W_k \sim \mathcal{CN}(0, N_0)$.

Hence, the transmitted SNR is:

$$\boxed{\text{Transmitted SNR} = \frac{E\{|X_k|^2\}}{Var\{W_k\}} = \frac{\mathcal{E}_s}{N_0}}$$

Note: $\mathcal{E}_s = 4\mathcal{E}_b$.

Received SNR of 16-QAM

After demodulation:

$$\tilde{Y}_k = X_k + \tilde{W}_k$$

where $\mathcal{E}_s = E\{|X_k|^2\}$ and $\tilde{W}_k \sim \mathcal{CN}(0, \frac{N_0}{|H_k|^2})$.

Hence, the received SNR is:

$$\text{Received SNR}_k = \frac{E\{|X_k|^2\}}{Var\{\tilde{W}_k\}} = \frac{E\{|X_k|^2\}}{N_0/|H_k|^2} = \frac{|H_k|^2\mathcal{E}_s}{N_0}$$

$$\text{Average Received SNR} = \frac{1}{N}\sum_{k=0}^{N-1}\frac{|H_k|^2\mathcal{E}_s}{N_0} = \frac{\mathcal{E}_s}{N_0}\frac{1}{N}\sum_{k=0}^{N-1}|H_k|^2$$

If $h[n]$ is normalized, then $\frac{1}{N}\sum_k |H[k]|^2 = \sum_n |h[n]|^2 = 1$. Therefore:

$$\boxed{\text{Average Received SNR} = \frac{\mathcal{E}_s}{N_0}}$$

SER of 4-PAM

For 4-PAM, the $\mathcal{E}_{avg}$ and SER are

$$\begin{aligned}
\mathcal{E}_s &= \frac{1}{4}\sum_{m=1}^{4} A_m^2 = \frac{1}{4}\left(\left(\frac{-3d_{min}}{2}\right)^2 + \left(\frac{-d_{min}}{2}\right)^2 + \left(\frac{d_{min}}{2}\right)^2 + \left(\frac{3d_{min}}{2}\right)^2\right) \\
&= \frac{5}{4}d_{min}^2
\end{aligned}$$

$$\begin{aligned}
SER = P_e &= \sum_{m=1}^{4} P(e|\vec{s}_i)P(\vec{s}_i) = \frac{1}{4}\sum_{m=1}^{4} P(e|\vec{s}_i) \\
&= \frac{2}{4}Q\left(\frac{d_{min}}{2\sigma}\right) + \frac{2}{4}2Q\left(\frac{d_{min}}{2\sigma}\right) \\
&= \frac{3}{2}Q\left(\frac{d_{min}}{2\sigma}\right) \\
&= \frac{3}{2}Q\left(\sqrt{\frac{\mathcal{E}_s}{5\sigma^2}}\right)
\end{aligned}$$

SER of 16-QAM

For 16-QAM, the $\mathcal{E}_s$, σ and SER are

$$\mathcal{E}_s = 2 \times \mathcal{E}_{s,4PAM} \quad \text{and} \quad \sigma^2 = 2 \times \sigma^2_{4PAM}$$

$$\begin{aligned}
SER = P_e = 1 - P_c = 1 - P^2_{c,4PAM} &= 1 - (1 - P_{e,4PAM})^2 \\
&= 1 - \left(1 - \frac{3}{2} Q\left(\sqrt{\frac{\mathcal{E}_s}{5\sigma^2}}\right)\right)^2 \\
&= 3Q\left(\sqrt{\frac{\mathcal{E}_s}{5\sigma^2}}\right) - \frac{9}{4} Q^2\left(\sqrt{\frac{\mathcal{E}_s}{5\sigma^2}}\right)
\end{aligned}$$

Our model is $\tilde{Y}_k = X_k + \tilde{W}_k$, where $\mathcal{E}_s = E\{|X_k|^2\}$ and $\tilde{W}_k \sim \mathcal{CN}(0, \frac{N_0}{|H_k|^2})$. Therefore:

$$SER_k = 3Q\left(\sqrt{\frac{|H_k|^2 \mathcal{E}_s}{5N_0}}\right) - \frac{9}{4} Q^2\left(\sqrt{\frac{|H_k|^2 \mathcal{E}_s}{5N_0}}\right)$$

BER of 16-QAM

The SER of standard 16-QAM is

$$P_{e,k} = 3Q\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right)$$

And for gray coding, $P_b \approx P_e/4$.
Hence, the BER of the kth symbol is

$$P_{b,k} \approx \frac{3}{4}Q\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right) - \frac{9}{16}Q^2\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right)$$

The average BER is

$$\boxed{BER = E\{P_{b,k}\} \approx E\left\{\frac{3}{4}Q\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right) - \frac{9}{16}Q^2\left(\sqrt{\frac{|H_k|^2\mathcal{E}_s}{5N_0}}\right)\right\}}$$

Appendix : BPSK Code 1

```
% Simulation baseband OFDM with BPSK modulation in staitc channel with AWGN
qfunc = @(x) 0.5 * erfc(x / sqrt(2));
rng(100);

%% Parameters
% BER to transmitted SNR_dB curve
SNR_TX_dB = 0:1:16; % Transmitted SNR
sim_pts = length(SNR_TX_dB);
BER_sim = zeros(1, sim_pts);
BER_theor = zeros(1, sim_pts);

% Transmitter
M = 64 * 1e5; % Transmit M bits, multiple of N
N = 64; % Number of subcarriers
mu = 16; % Length of CP
Es = 1; % Transmitted symbol energy

num_ofdm_symbols = M / N; % Number of ofdm symbols

% Channel
L = 5; % Length of channel
h = 0.5 .^ (0:L-1) * (1 + 1j);% Static channel (LTI), complex
h = h / norm(h); % Normalization
H = fft(h, N); % Frequency response

% Receiver
offset = 0; % Different fft window
k = (0:N-1).'; % Frequency index
```

Appendix : BPSK Code 2

```
%% Loop for different SNR_dB value
for i = 1:sim_pts
    SNR_TX = 10^(SNR_TX_dB(i) / 10); % SNR_dB = 10log(SNR)

    %% Transmitter
    b = randi([0, 1], 1, M); % Serial bits
    bpsk = 2 * b - 1; % BPSK modulation
    X = reshape(bpsk, N, num_ofdm_symbols); % S/P = construct a matrix X in which every column is a fr
    x = ifft(X, N) * sqrt(N); % IDFT
    x_cp = [x(end - mu + 1 : end, :); x]; % Add CP
    x_cp_serial = x_cp(:); % P/S

    %% Channel
    r_LTI = filter(h,1,x_cp_serial);

    % AWGN
    N0 = Es / SNR_TX; % SNR_TX = Es / N0
    sigma2 = N0 / 2; % Var of noise for Re & Im
    w = sqrt(sigma2) * (randn(size(r_LTI)) + 1j * randn(size(r_LTI))); % Complex Gaussian white noise
```

Appendix : BPSK Code 3

```
    %% Receiver
    r_serial = r_LTI + w; % Equivalent baseband received signal
    r = reshape(r_serial, N + mu, num_ofdm_symbols); % P/S
    y = r(mu + 1 - offset : end - offset, :); % Minus CP
    Y = fft(y, N) / sqrt(N); % DFT
    Y = Y .* exp(1j*2*pi*k*offset/N); % Recover the phase from time shifting
    Y_eq = real((conj(H) ./ abs(H)).' .* Y); % Equalization
    b_hat = (Y_eq > 0); % Decesion
    b_hat = b_hat(:).'; % P/S

    %% Error
    total_error = sum(b_hat ~= b);

    %% BER
    % Theoretical BER
    BER_k = qfunc(sqrt(2 * abs(H).^2 * Es / N0)); % BER for each subcarrier
    BER_theor(i) = mean(BER_k); % Average BER

    % Simulated BER
    BER_sim(i) = total_error / M;
end

%% Plot
```

Appendix : 16QAM Code 1

```
% Simulation baseband OFDM with 16QAM modulation in staitc channel with AWGN
qfunc = @(x) 0.5 * erfc(x / sqrt(2));
rng(100);

%% Parameters
% BER & SER to SNR_dB curve
SNR_dB = 0:1:22; % SNR = Es/N0
sim_pts = length(SNR_dB);
BER_sim = zeros(1, sim_pts);
BER_theor = zeros(1, sim_pts);
SER_sim = zeros(1, sim_pts);
SER_theor = zeros(1, sim_pts);

% Transmitter
M = 64 * 1e5; % Transmit M bits, multiple of N*k_QAM
N = 64; % Number of subcarriers
mu = 16; % Length of CP
Es = 10; % Transmitted average symbol energy
M_QAM = 16; % 16-QAM
k_QAM = 4; % bits per symbol, log_2(16)=4

num_ofdm_symbols = M / (N * k_QAM); % Number of ofdm symbols

% Channel
L = 5; % Length of channel
h = 0.5 .^ (0:L-1) * (1 + 1j);% Static channel (LTI), complex
h = h / norm(h); % Normalization
H = fft(h, N); % Frequency response

% Receiver
offset = 0; % Different fft window
k = (0:N-1).'; % Frequency index
```

Appendix : 16QAM Code 2

```
%% Loop for different SNR_dB value
for i = 1:sim_pts
    SNR = 10^(SNR_dB(i) / 10); % SNR_dB = 10log(SNR)

    %% Transmitter
    b = randi([0, 1], 1, M); % Serial bits
    qam16 = qammod(b.', M_QAM, 'gray', 'InputType','bit', 'UnitAveragePower', true); % 16QAM modulation
    qam16 = qam16 * sqrt(Es); % Average symbol energy = Es
    X = reshape(qam16, N, num_ofdm_symbols); % S/P = construct a matrix X in which every column is a freq domain OFDM symbol
    x = ifft(X, N) * sqrt(N); % IDFT
    x_cp = [x(end - mu + 1 : end, :); x]; % Add CP
    x_cp_serial = x_cp(:); % P/S

    %% Channel
    r_LTI = filter(h,1,x_cp_serial);

    % AWGN
    N0 = Es / SNR; % SNR = Es / N0
    sigma2 = N0 / 2; % Var of noise for Re & Im
    w = sqrt(sigma2) * (randn(size(r_LTI)) + 1j * randn(size(r_LTI))); % Complex Gaussian white noise

    %% Receiver
    r_serial = r_LTI + w; % Equivalent baseband received signal
    r = reshape(r_serial, N + mu, num_ofdm_symbols); % S/P
    y = r(mu + 1 - offset : end - offset, :); % Minus CP
    Y = fft(y, N) / sqrt(N); % DFT
    Y = Y .* exp(1j*2*pi*k*offset/N); % Recover the phase from time shifting
    Y_eq = Y ./ H.'; % Equalization
    b_hat = qamdemod(Y_eq, M_QAM, 'gray', 'OutputType', 'bit', 'UnitAveragePower', false); % Decesion
    b_hat = b_hat(:).'; % P/S
```

Appendix : 16QAM Code 3

```
    %% BER
    % Simulated BER
    total_bit_error = sum(b_hat ~= b);
    BER_sim(i) = total_bit_error / M;

    % Theoretical BER
    Q = qfunc(sqrt(abs(H).^2 * Es / N0 / 5));
    SER_k = 3 * Q - 9/4 * Q.^2; % SER for each subcarrier
    BER_k = SER_k / 4; % For gray coding, Pb ~= Pe/4
    BER_theor(i) = mean(BER_k); % Average BER

    %% SER
    b_sym = qamdemod(qam16, M_QAM, 'gray', 'OutputType', 'integer', 'UnitAveragePower', false);
    b_sym_hat = qamdemod(Y_eq, M_QAM, 'gray', 'OutputType', 'integer', 'UnitAveragePower', false);
    b_sym = b_sym(:);
    b_sym_hat = b_sym_hat(:);

    % Simulated SER
    total_symbol_error = sum(b_sym ~= b_sym_hat);
    SER_sim(i) = total_symbol_error / (M/4);

    % Theoretical BER
    SER_theor(i) = mean(SER_k); % Average SER
end

%% Plot
```

The End